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Dual Concepts of Almost Distance-Regularity and the Spectral Excess Theorem*

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Abstract

Generally speaking, ‘almost distance-regular’ graphs share some, but not necessarily all, of the regularity properties that characterize distance-regular graphs. In this paper we propose two new dual concepts of almost distance-regularity, thus giving a better understanding of the properties of distance-regular graphs. More precisely, we characterize m -partially distance-regular graphs and j -punctually eigenspace distance-regular graphs by using their spectra. Our results can also be seen as a generalization of the so-called spectral excess theorem for distance-regular graphs, and they lead to a dual version of it.

Keywords: Distance-regular graph, Distance matrices, Eigenvalues, Idempotents, Local spectrum, Predistance polynomials

2010 Mathematics Subject Classification: 05E30, 05C50

1 Preliminaries

Almost distance-regular graphs, recently studied in the literature, are graphs which share some, but not necessarily all, of the regularity properties that characterize distance-regular graphs. Two examples of the former are partially distance-regular graphs [14] and m -walk-regular graphs [6].

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In this paper we propose and characterize two dual concepts of almost distance-regularity, and study some cases where distance-regularity is attained. As in the theory of distance-regular graphs, the two proposed concepts lead to several duality results. Our results can also be seen as a generalization of the so-called spectral excess theorem for distance-regular graphs (see [9]; for short proofs, see [15, 10]). This theorem characterizes distance-regular graphs by their spectra and the average number of vertices at extremal distance. A dual version of this theorem is also derived.

We use standard concepts and results for distance-regular graphs [1, 2], spectral graph theory [4, 12], and spectral and algebraic characterizations of distance-regular graphs [8]. Moreover, for some more details and other concepts of almost distance-regularity (such as distance-polynomial and partially distance-regular graphs), we refer the reader to our recent paper [5]. In what follows, we recall the main concepts, terminology, and results involved.

Let Γ be a simple, connected, δ -regular graph, with vertex set V , order $n = |V|$, and adjacency matrix \mathbf{A} . The *distance* between two vertices u and v is denoted by $\text{dist}(u, v)$, so the *diameter* of Γ is $D = \max_{u, v \in V} \text{dist}(u, v)$. The set of vertices at distance i from a given vertex $u \in V$ is denoted by $\Gamma_i(u)$, for $i = 0, 1, \dots, D$. The *distance- i graph* Γ_i is the graph with vertex set V and where two vertices u and v are adjacent if and only if $\text{dist}(u, v) = i$ in Γ . Its adjacency matrix \mathbf{A}_i is usually referred to as the *distance- i matrix* of Γ . The spectrum of Γ is denoted by $\text{sp } \Gamma = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \dots, \lambda_d^{m_d}\}$, where the different eigenvalues of Γ are in decreasing order, $\lambda_0 > \lambda_1 > \dots > \lambda_d$, and the superscripts stand for their multiplicities $m_i = m(\lambda_i)$.

1.1 The predistance and preidempotent polynomials

From the spectrum of Γ , we consider the *predistance polynomials* $\{p_i\}_{0 \leq i \leq d}$ which are orthogonal with respect to the following scalar product in $\mathbb{R}_d[x]$:

$$\langle f, g \rangle_\Delta = \frac{1}{n} \text{tr}(f(\mathbf{A})g(\mathbf{A})) = \frac{1}{n} \sum_{i=0}^d m_i f(\lambda_i) g(\lambda_i), \quad (1)$$

and which satisfy $\deg p_i = i$ and $\langle p_i, p_j \rangle_\Delta = \delta_{ij} p_i(\lambda_0)$, for all $i, j = 0, 1, \dots, d$. For more details, see [9]. Like every sequence of orthogonal polynomials, the predistance polynomials satisfy a three-term recurrence of the form

$$xp_i = \beta_{i-1} p_{i-1} + \alpha_i p_i + \gamma_{i+1} p_{i+1}, \quad i = 0, 1, \dots, d, \quad (2)$$

with $\beta_{-1} = \gamma_{d+1} = 0$. Some basic properties of these coefficients, such as $\alpha_i + \beta_i + \gamma_i = \lambda_0$ for $i = 0, 1, \dots, d$, and $\beta_i n_i = \gamma_{i+1} n_{i+1} \neq 0$ for

$i = 0, 1, \dots, d-1$, where $n_i = \|p_i\|_{\Delta}^2 = p_i(\lambda_0)$, can be found in [3]. Let ω_i be the leading coefficient of p_i . Then, from the above recurrence and since $p(0) = 1$, it is immediate that $\omega_i = (\gamma_1 \gamma_2 \cdots \gamma_i)^{-1}$ for $i = 1, \dots, d$.

For any graph, the sum of all the predistance polynomials gives the *Hoffman polynomial* H satisfying $H(\lambda_i) = n\delta_{0i}$, $i = 0, 1, \dots, d$, which characterizes regular graphs via the condition $H(\mathbf{A}) = \mathbf{J}$, the all-1 matrix [13]. Note that the leading coefficient ω_d of H (and also of p_d) is $\omega_d = n/\pi_0$.

From the predistance polynomials, we define the so-called *preidempotent polynomials* q_j , $j = 0, 1, \dots, d$, by

$$q_j(\lambda_i) = \frac{m_j}{n_i} p_i(\lambda_j), \quad i = 0, 1, \dots, d,$$

which are orthogonal with respect to the scalar product

$$\langle f, g \rangle_{\mathbf{A}} = \frac{1}{n} \text{tr}(f\{\mathbf{A}\}g\{\mathbf{A}\}) = \frac{1}{n} \sum_{i=0}^d n_i f(\lambda_i) g(\lambda_i), \quad (3)$$

where $f\{\mathbf{A}\} = \frac{1}{\sqrt{n}} \sum_{i=0}^d f(\lambda_i) p_i(\mathbf{A})$. Note that, since $q_j(\lambda_0) = m_j$, the duality between the two scalar products (1) and (3) and their associated polynomials is made apparent by writing

$$\langle p_i, p_j \rangle_{\Delta} = \frac{1}{n} \sum_{l=0}^d m_l p_i(\lambda_l) p_j(\lambda_l) = \delta_{ij} n_i, \quad i, j = 0, 1, \dots, d, \quad (4)$$

$$\langle q_i, q_j \rangle_{\mathbf{A}} = \frac{1}{n} \sum_{l=0}^d n_l q_i(\lambda_l) q_j(\lambda_l) = \delta_{ij} m_i, \quad i, j = 0, 1, \dots, d. \quad (5)$$

1.2 Vector spaces, algebras and bases

Let Γ be a graph with diameter D , adjacency matrix \mathbf{A} and $d+1$ distinct eigenvalues. We consider the vector spaces $\mathcal{A} = \mathbb{R}_d[\mathbf{A}] = \text{span}\{\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \dots, \mathbf{A}^d\}$ and $\mathcal{D} = \text{span}\{\mathbf{I}, \mathbf{A}, \mathbf{A}_2, \dots, \mathbf{A}_D\}$, with dimensions $d+1$ and $D+1$, respectively. Then, \mathcal{A} is an algebra with the ordinary product of matrices, known as the *adjacency algebra*, with orthogonal bases $A_p = \{p_0(\mathbf{A}), p_1(\mathbf{A}), p_2(\mathbf{A}), \dots, p_d(\mathbf{A})\}$ and $A_{\lambda} = \{\mathbf{E}_0, \mathbf{E}_1, \dots, \mathbf{E}_d\}$, where the matrices \mathbf{E}_i , $i = 0, 1, \dots, d$, corresponding to the orthogonal projections onto the eigenspaces, are the (*principal*) *idempotents* of \mathbf{A} . Besides, since $\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \dots, \mathbf{A}^D$ are linearly independent, we have that $\dim \mathcal{A} = d+1 \geq D+1$ and, therefore, we always have $D \leq d$ [1]. Moreover, \mathcal{D} forms an algebra with the entrywise or Hadamard product of matrices, defined by

$(\mathbf{X} \circ \mathbf{Y})_{uv} = \mathbf{X}_{uv} \mathbf{Y}_{uv}$. We call \mathcal{D} the *distance \circ -algebra*, which has orthogonal basis $D_\lambda = \{\mathbf{I}, \mathbf{A}, \mathbf{A}_2, \dots, \mathbf{A}_d\}$.

From now on, we work with the vector space $\mathcal{T} = \mathcal{A} + \mathcal{D}$, and relate the distance- i matrices $\mathbf{A}_i \in \mathcal{D}$ to the matrices $p_i(\mathbf{A}) \in \mathcal{A}$. Note that \mathbf{I} , \mathbf{A} , and \mathbf{J} are matrices in $\mathcal{A} \cap \mathcal{D}$ since $\mathbf{J} = H(\mathbf{A}) \in \mathcal{A}$. Recall that $\mathcal{A} = \mathcal{D}$ if and only if Γ is distance-regular (see [1, 2]). In this case, we have $D = d$, and the predistance polynomials become the *distance polynomials* satisfying $\mathbf{A}_i = p_i(\mathbf{A})$. In \mathcal{T} , we consider the following scalar product:

$$\langle \mathbf{R}, \mathbf{S} \rangle = \frac{1}{n} \text{tr}(\mathbf{R}\mathbf{S}) = \frac{1}{n} \text{sum}(\mathbf{R} \circ \mathbf{S}), \quad (6)$$

where $\text{sum}(\mathbf{M})$ denotes the sum of all entries of \mathbf{M} . Observe that the factor $1/n$ assures that $\|\mathbf{I}\|^2 = 1$, whereas $\|\mathbf{J}\|^2 = n$. Note also that the *average degree* of Γ_i is $\bar{\delta}_i = \|\mathbf{A}_i\|^2$ and the *average multiplicity* of λ_j is $\bar{m}_j = \frac{m_j}{n} = \|\mathbf{E}_j\|^2$. According to (1), this scalar product of matrices satisfies $\langle f(\mathbf{A}), g(\mathbf{A}) \rangle = \langle f, g \rangle_\Delta$.

2 Two dual approaches to almost distance-regularity

Here we limit ourselves to the case of graphs with spectrally maximum diameter (or the ‘non-degenerate’ case) $D = d$. Consequently, we will use indiscriminately the two symbols, D and d , depending on what we are referring to. In this context, let us consider the following two definitions of almost distance-regularity:

Definition 2.1 *For a given i , $0 \leq i \leq D$, a graph Γ is i -punctually distance-regular when there exist constants p_{ji} such that*

$$\mathbf{A}_i \mathbf{E}_j = p_{ji} \mathbf{E}_j \quad (7)$$

for every $j = 0, 1, \dots, d$; and Γ is m -partially distance-regular when it is i -punctually distance-regular for all $i \leq m$.

Definition 2.2 *For a given j , $0 \leq j \leq d$, a graph Γ is j -punctually eigenspace distance-regular when there exist constants q_{ij} such that*

$$\mathbf{E}_j \circ \mathbf{A}_i = q_{ij} \mathbf{A}_i \quad (8)$$

for every $i = 0, 1, \dots, D$; and Γ is m -partially eigenspace distance-regular when it is j -punctually eigenspace distance-regular for all $j \leq m$.

Notice that the concepts of D -partial distance-regularity and d -partial eigenspace distance-regularity coincide with the known dual definitions of distance-regularity (see [2]).

Some basic characterizations of punctual distance-regularity, in terms of the distance matrices and the idempotents, were given in [5].

Proposition 2.3 ([5]) *Let $D = d$. Then, Γ is i -punctually distance-regular if and only if any of the following conditions holds:*

- (a1) $\mathbf{A}_i \in \mathcal{A}$,
- (a2) $p_i(\mathbf{A}) \in \mathcal{D}$,
- (a3) $\mathbf{A}_i = p_i(\mathbf{A})$.

Following the duality between Definitions 2.1 and 2.2, it seems natural to conjecture the dual of this proposition: A graph Γ is j -punctually eigenspace distance-regular if and only if any of the following conditions is satisfied:

- (b1) $\mathbf{E}_j \in \mathcal{D}$,
- (b2) $q_j[\mathbf{A}] \in \mathcal{A}$,
- (b3) $\mathbf{E}_j = q_j[\mathbf{A}]$,

where $f[\mathbf{A}] = \frac{1}{n} \sum_{i=0}^d f(\lambda_i) \mathbf{A}_i$. However, although (b1) is clearly equivalent to Definition 2.2 and (b3) \Rightarrow (b1), (b2), until now we have not been able to prove any of the other equivalences and we leave them as conjectures.

In order to derive some new characterizations of punctual distance-regularity, besides the already defined $\bar{\delta}_i$ and \bar{m}_j , we consider the following average numbers:

- The *average crossed local multiplicities* are

$$\bar{m}_{ij} = \frac{1}{n\bar{\delta}_i} \sum_{\text{dist}(u,v)=i} m_{uv}(\lambda_j) = \frac{\langle \mathbf{E}_j, \mathbf{A}_i \rangle}{\|\mathbf{A}_i\|^2}, \quad (9)$$

where $m_{uv}(\lambda_j) = (\mathbf{E}_j)_{uv}$ are the *crossed local multiplicities*.

- The *average number of shortest i -paths from a vertex* is

$$\bar{P}_i = \frac{1}{n} \sum_{u \in V} P_i(u) = \frac{1}{n} \text{sum}(\mathbf{A}^i \circ \mathbf{A}_i) = \langle \mathbf{A}^i, \mathbf{A}_i \rangle = \frac{1}{\omega_i} \langle p_i(\mathbf{A}), \mathbf{A}_i \rangle, \quad (10)$$

where $P_i(u)$ denotes the number of shortest paths from a vertex u to the vertices in $\Gamma_i(u)$ and $\omega_i = (\gamma_1\gamma_2\cdots\gamma_i)^{-1}$ is the leading coefficient of p_i , $i = 1, \dots, d$.

- The average number of shortest i -paths is

$$\bar{a}_i^{(i)} = \frac{1}{n\bar{\delta}_i} \text{sum}(\mathbf{A}^i \circ \mathbf{A}_i) = \frac{\bar{P}_i}{\bar{\delta}_i}. \quad (11)$$

Proposition 2.4 *Let Γ be a graph with predistance polynomials p_i and recurrence coefficients $\gamma_i, \alpha_i, \beta_i$, $i = 0, 1, \dots, d$. Then, Γ is i -punctually distance-regular if and only if any of the following equalities holds:*

$$(a1) \quad \frac{1}{\bar{\delta}_i} = \sum_{j=0}^d \frac{\bar{m}_{ij}^2}{\bar{m}_j}.$$

$$(a2) \quad \bar{P}_i = \frac{1}{\omega_i} \sqrt{p_i(\lambda_0)\bar{\delta}_i} = \sqrt{\beta_0\beta_1\cdots\beta_{i-1}\bar{\delta}_i\gamma_i\gamma_{i-1}\cdots\gamma_1}.$$

$$(a3) \quad \omega_i \bar{a}_i^{(i)} = 1 \quad \text{and} \quad \bar{\delta}_i = p_i(\lambda_0).$$

Moreover, Γ is j -punctually eigenspace distance-regular if and only if

$$(b1) \quad \bar{m}_j = \sum_{i=0}^D \bar{\delta}_i \bar{m}_{ij}^2.$$

Proof. (a1) This is a result from [5].

(a2) From (10) and the Cauchy-Schwarz inequality, we get

$$\omega_i \bar{P}_i = \langle p_i(\mathbf{A}), \mathbf{A}_i \rangle \leq \|p_i(\mathbf{A})\| \|\mathbf{A}_i\| = \sqrt{p_i(\lambda_0)\bar{\delta}_i} = \sqrt{\frac{\beta_0\beta_1\cdots\beta_{i-1}}{\gamma_1\gamma_2\cdots\gamma_i}\bar{\delta}_i}. \quad (12)$$

Moreover, equality occurs if and only if the matrices $p_i(\mathbf{A})$ and \mathbf{A}_i are proportional, which is equivalent to Γ being i -punctually distance-regular by Proposition 2.3.

(a3) From (11) and (12) we have that $\omega_i \bar{a}_i^{(i)} \leq \sqrt{p_i(\lambda_0)/\bar{\delta}_i}$, with equality if and only if Γ is i -punctually distance-regular. Thus, if the conditions in (a3) hold, Γ satisfies the claimed property. Conversely, if Γ is i -punctually distance-regular, both equalities in (a3) are simple consequences of $p_i(\mathbf{A}) = \mathbf{A}_i$. Indeed, the first one comes from considering the uv -entries, with $\text{dist}(u, v) = i$, in the above matrix equation, whereas the second one is obtained by taking square norms.

(b1) From (9), we find that the orthogonal projection of \mathbf{E}_j on \mathcal{D} is $\widehat{\mathbf{E}}_j = \sum_{i=0}^D \bar{m}_{ij} \mathbf{A}_i$. Now, from $\|\widehat{\mathbf{E}}_j\|^2 \leq \|\mathbf{E}_j\|^2$ we get

$$\sum_{i=0}^D \bar{m}_{ij}^2 \|\mathbf{A}_i\|^2 = \sum_{i=0}^D \bar{\delta}_i \bar{m}_{ij}^2 \leq \bar{m}_j$$

and, in the case of equality, Definition 2.2 applies with $q_{ij} = \bar{m}_{ij}$. \square

Notice the duality between (a1) and (b1) with $\frac{1}{\delta_i}$ and \bar{m}_j .

Now, let us consider the more global concept of partial distance-regularity. In this case, we also have the following new result where, for a given $0 \leq i \leq d$, $s_i = \sum_{j=0}^i p_j$, $t_i = H - s_{i-1} = \sum_{j=i}^d p_j$, $\mathbf{S}_i = \sum_{j=0}^i \mathbf{A}_j$, and $\mathbf{T}_i = \mathbf{J} - \mathbf{S}_{i-1} = \sum_{j=i}^d \mathbf{A}_j$.

Proposition 2.5 *A graph Γ is m -partially distance-regular if and only if any of the following conditions holds:*

- (a1) Γ is i -punctually distance-regular for $i = m, m-1, \dots, \max\{2, 2m-d\}$.
- (a2) Γ is m -punctually distance-regular and $t_{m+1}(\mathbf{A}) \circ \mathbf{S}_m = \mathbf{O}$.
- (a3) $s_i(\mathbf{A}) = \mathbf{S}_i$ for $i = m, m-1$.

Proof. In all cases, the necessity is clear since $p_i(\mathbf{A}) = \mathbf{A}_i$ for every $0 \leq i \leq m$ (for (a2), note that $t_{m+1}(\mathbf{A}) = \mathbf{J} - \mathbf{S}_m(\mathbf{A})$). Then, let us prove sufficiency. The result in (a1) is basically Proposition 3.7 in [5]. In order to prove (a2), we show by (backward) induction that $p_i(\mathbf{A}) = \mathbf{A}_i$ and $t_{i+1}(\mathbf{A}) \circ \mathbf{S}_i = \mathbf{O}$ for $i = m, m-1, \dots, 0$. By assumption, these equations are valid for $i = m$. Suppose now that $p_i(\mathbf{A}) = \mathbf{A}_i$ and $t_{i+1}(\mathbf{A}) \circ \mathbf{S}_i = \mathbf{O}$ for some $i > 0$. Then, $t_i(\mathbf{A}) \circ \mathbf{S}_i = \mathbf{A}_i$ and, multiplying both terms by \mathbf{S}_{i-1} (with the Hadamard product), we get $t_i(\mathbf{A}) \circ \mathbf{S}_{i-1} = \mathbf{O}$. So, what remains is to show that $p_{i-1}(\mathbf{A}) = \mathbf{A}_{i-1}$. To this end, let us consider the following three cases:

- (i) For $\text{dist}(u, v) > i-1$, we have $(p_{i-1}(\mathbf{A}))_{uv} = 0$.
- (ii) For $\text{dist}(u, v) = i-1$, we have $(t_{i+1}(\mathbf{A}))_{uv} = 0$, so $(p_{i-1}(\mathbf{A}))_{uv} = (s_{i-1}(\mathbf{A}))_{uv} = (s_{i-1}(\mathbf{A}))_{uv} + (\mathbf{A}_i)_{uv} = (s_i(\mathbf{A}))_{uv} = 1 - (t_{i+1}(\mathbf{A}))_{uv} = 1$.
- (iii) For $\text{dist}(u, v) < i-1$, we use the recurrence (2) to write

$$xp_i = \sum_{j=i}^d xp_j = \sum_{j=i}^d (\beta_{j-1}p_{j-1} + \alpha_j p_j + \gamma_{j+1}p_{j+1})$$

$$\begin{aligned}
&= \beta_{i-1}p_{i-1} - \gamma_i p_i + \sum_{j=i}^d (\alpha_j + \beta_j + \gamma_j)p_j \\
&= \beta_{i-1}p_{i-1} - \gamma_i p_i + \delta t_i,
\end{aligned}$$

which gives

$$\mathbf{A}t_i(\mathbf{A}) = \beta_{i-1}p_{i-1}(\mathbf{A}) - \gamma_i \mathbf{A}_i + \delta t_i(\mathbf{A}).$$

Then, since $(t_i(\mathbf{A}))_{uv} = (\mathbf{A}_i)_{uv} = 0$ and $\beta_{i-1} \neq 0$, we get

$$(p_{i-1}(\mathbf{A}))_{uv} = \frac{1}{\beta_{i-1}} (\mathbf{A}t_i(\mathbf{A}))_{uv} = \frac{1}{\beta_{i-1}} \sum_{w \in \Gamma(u)} (t_i(\mathbf{A}))_{wv} = 0,$$

because $\text{dist}(v, w) \leq \text{dist}(v, u) + \text{dist}(u, w) \leq i - 1$ for the relevant w .

From (i), (ii), and (iii), we have that $p_{i-1}(\mathbf{A}) = \mathbf{A}_{i-1}$, so by induction Γ is m -partially distance-regular, and the sufficiency of (a2) is proven. Finally, the sufficiency of (a3) follows from that of (a2) because $s_i(\mathbf{A}) = \mathbf{S}_i$ for every $i \in \{m-1, m\}$ implies that $p_m(\mathbf{A}) = (s_m - s_{m-1})(\mathbf{A}) = \mathbf{S}_m - \mathbf{S}_{m-1} = \mathbf{A}_m$ and $t_{m+1}(\mathbf{A}) \circ \mathbf{S}_m = (\mathbf{J} - s_m(\mathbf{A})) \circ \mathbf{S}_m = (\mathbf{J} - \mathbf{S}_m) \circ \mathbf{S}_m = \mathbf{O}$. \square

Given some vertex u and an integer $i \leq \text{ecc}(u)$, we denote by $N_i(u)$ the i -neighborhood of u , which is the set of vertices that are at distance at most i from u . In [8] it was proved that $s_i(\lambda_0)$ is upper bounded by the harmonic mean of the numbers $|N_i(u)|$ and equality is attained if and only if $s_i(\mathbf{A}) = \mathbf{S}_i$. A direct consequence of this property and Proposition 2.5(a3) is the following characterization.

Theorem 2.6 *A graph Γ is m -partially distance-regular if and only if, for every $i \in \{m-1, m\}$,*

$$s_i(\lambda_0) = \frac{n}{\sum_{u \in V} |N_i(u)|^{-1}}.$$

3 Distance-regular graphs

Let us particularize our results to the case of distance-regular graphs. With this aim, we use the following theorem giving some known characterizations.

Theorem 3.1 ([7, 11]) *A graph Γ with $d+1$ distinct eigenvalues and diameter $D = d$ is distance-regular if and only if any of the following statements is satisfied:*

- (a) Γ is D -punctually distance-regular.
- (b) Γ is j -punctually eigenspace distance-regular for $j = 1, d$.

In fact, notice that (a) corresponds to any of the conditions in Proposition 2.5 with $m = d$. Moreover, the duality between (a) and (b) is made apparent when they are stated as follows:

- (a) $\mathbf{A}_0(= \mathbf{I}), \mathbf{A}_1(= \mathbf{A}), \mathbf{A}_D \in \mathcal{A}$;
- (b) $\mathbf{E}_0(= \frac{1}{n}\mathbf{J}), \mathbf{E}_1, \mathbf{E}_d \in \mathcal{D}$.

Then, by using Theorem 3.1 and Proposition 2.4(a1) and (b1), and Theorem 2.6 (with $m = d$), we have the spectral excess theorem [9] in the next condition (a), its dual form in (b), and its harmonic mean version [8, 15] in (c).

Theorem 3.2 *A regular graph Γ with $D = d$ is distance-regular if and only if any of the following equalities holds:*

- (a) $\frac{1}{\bar{\delta}_d} = \sum_{j=0}^d \frac{\bar{m}_{dj}^2}{\bar{m}_j}$.
- (b) $\bar{m}_j = \sum_{i=0}^D \bar{\delta}_i \bar{m}_{ij}^2$ for $j = 1, d$.
- (c) $s_{d-1}(\lambda_0) = \frac{n}{\sum_{u \in V} |N_{d-1}(u)|^{-1}}$.

In fact, condition (a) is usually written in its equivalent form $\bar{\delta}_d = p_d(\lambda_0)$ as, when $i = d$, the first condition in Proposition 2.4(a.3) always holds since

$$\bar{a}_d^{(d)} = \frac{1}{\bar{\delta}_d} \langle \mathbf{A}^d, \mathbf{A}_d \rangle = \frac{1}{\bar{\delta}_d \omega_d} \langle H(\mathbf{A}), \mathbf{A}_d \rangle = \frac{1}{\bar{\delta}_d \omega_d} \langle \mathbf{J}, \mathbf{A}_d \rangle = \frac{1}{\bar{\delta}_d \omega_d} \|\mathbf{A}_d\|^2 = \frac{1}{\omega_d}.$$

Notice also that, in (c), we do not need to impose the condition of Theorem 2.6 for $i = d$ since $s_d(\lambda_0) = H(\lambda_0) = N_d(u) = n$ for every $u \in V$.

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